

Necessary Optimality Condition for a Discrete Dead Oil Isotherm Optimal Control Problem

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Summary. We obtain necessary optimality conditions for a semi-discretized optimal control problem for the classical system of nonlinear partial differential equations modelling the water-oil (isothermal dead-oil model).

Key words: extraction of hydrocarbons; dead oil isotherm problem; optimality conditions.

1 Introduction

We study an optimal control problem in the discrete case whose control system is given by the following system of nonlinear partial differential equations,

$$\begin{cases} \partial_t u - \Delta \varphi(u) = \operatorname{div}(g(u)\nabla p) & \text{in } Q_T = \Omega \times (0, T), \\ \partial_t p - \operatorname{div}(d(u)\nabla p) = f & \text{in } Q_T = \Omega \times (0, T), \\ u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0, \\ p|_{\partial\Omega} = 0, \quad p|_{t=0} = p_0, \end{cases} \quad (1)$$

which result from a well established model for oil engineering within the framework of the mechanics of a continuous medium [3]. The domain Ω is an open bounded set in \mathbb{R}^2 with a sufficiently smooth boundary. Further hypotheses on the data of the problem will be specified later.

At the time of the first run of a layer, the flow of the crude oil towards the surface is due to the energy stored in the gases under pressure in the natural hydraulic system. To mitigate the consecutive decline of production and the decomposition of the site, water injections are carried out, well before the normal exhaustion of the layer. The water is injected through wells with high pressure, by pumps specially drilled to this end. The pumps allow the displacement of the crude oil towards the wells of production. More precisely, the problem consists in seeking the admissible control parameters which minimize

a certain objective functional. In our problem, the main goal is to distribute properly the wells in order to have the best extraction of the hydrocarbons. For this reason, we consider a cost functional containing different parameters arising in the process. To address the optimal control problem, we use the Lagrangian method to derive an optimality system: from the cost function we introduce a Lagrangian; then, we calculate the Gâteaux derivative of the Lagrangian with respect to its variables. This technique was used, in particular, by A. Masserey et al. for electromagnetic models of induction heating [1, 7], and by H.-C. Lee and T. Shilkin for the thermistor problem [5].

We consider the following cost functional:

$$J(u, p, f) = \frac{1}{2} \|u - U\|_{2, Q_T}^2 + \frac{1}{2} \|p - P\|_{2, Q_T}^2 + \frac{\beta_1}{2} \|f\|_{2q_0, Q_T}^{2q_0} + \frac{\beta_2}{2} \|\partial_t f\|_{2, Q_T}^2. \quad (2)$$

The control parameters are the reduced saturation of oil u , the pressure p , and f . The coefficients $\beta_1 > 0$ and $\beta_2 \geq 0$ are two coefficients of penalization, and $q_0 > 1$. The first two terms in (2) allow to minimize the difference between the reduced saturation of oil u , the global pressure p and the given data U and P . The third and fourth terms are used to improve the quality of exploitation of the crude oil. We take $\beta_2 = 0$ just for the sake of simplicity. It is important to emphasize that our choice of the cost function is not unique. One can always add additional terms of penalization to take into account other properties which one may wish to control. Recently, we proved in [8] results of existence, uniqueness, and regularity of the optimal solutions to the problem of minimizing (2) subject to (1), using the theory of parabolic problems [4, 6]. Here, our goal is to obtain necessary optimality conditions which may be easily implemented on a computer. More precisely, we address the problem of obtaining necessary optimality conditions for the semi-discretized time problem.

In order to be able to solve problem (1)-(2) numerically, we use discretization of the problem in time by a method of finite differences. For a fixed real N , let $\tau = \frac{T}{N}$ be the step of a uniform partition of the interval $[0, T]$ and $t_n = n\tau$, $n = 1, \dots, N$. We denote by u^n an approximation of u . The discrete cost functional is then defined as follows:

$$J(u^n, p^n, f^n) = \frac{\tau}{2} \sum_{n=1}^N \int_{\Omega} \{ \|u^n - U\|_{2, \Omega}^2 + \|p^n - P\|_{2, \Omega}^2 + \beta_1 \|f^n\|_{2q_0, \Omega}^{2q_0} \} dx. \quad (3)$$

It is now possible to state our optimal control problem: find $(\bar{u}^n, \bar{p}^n, \bar{f}^n)$ which minimizes (3) among all functions (u^n, p^n, f^n) satisfying

$$\begin{cases} \frac{u^{n+1} - u^n}{\tau} - \Delta \varphi(u^n) = \operatorname{div}(g(u^n) \nabla p) & \text{in } \Omega, \\ \frac{p^{n+1} - p^n}{\tau} - \operatorname{div}(d(u^n) \nabla p^n) = f^n & \text{in } \Omega, \\ u|_{\partial \Omega} = 0, \quad u|_{t=0} = u_0, \\ p|_{\partial \Omega} = 0, \quad p|_{t=0} = p_0. \end{cases} \quad (4)$$

The soughtafter necessary optimality conditions are proved in §3 under suitable hypotheses on the data of the problem.

2 Notation, hypotheses, and functional spaces

Our main objective is to obtain necessary conditions for a triple $(\bar{u}^n, \bar{p}^n, \bar{f}^n)$ to minimize (3) among all the functions (u^n, p^n, f^n) verifying (4). In the sequel we assume that φ , g and d are real valued functions, respectively of class C^3 , C^2 and C^1 , satisfying:

- (H1) $0 < c_1 \leq d(r)$, $\varphi(r) \leq c_2$; $|d'(r)|$, $|\varphi'(r)|$, $|\varphi''(r)| \leq c_3 \quad \forall r \in \mathbb{R}$.
(H2) $u_0, p_0 \in C^2(\bar{\Omega})$, and $U, P \in L^2(\Omega)$, where $u_0, p_0, U, P : \Omega \rightarrow \mathbb{R}$, and $u_0|_{\partial\Omega} = p_0|_{\partial\Omega} = 0$.

We consider the following spaces:

$$W_p^1(\Omega) := \{u \in L^p(\Omega), \nabla u \in L^p(\Omega)\},$$

endowed with the norm $\|u\|_{W_p^1(\Omega)} = \|u\|_{p,\Omega} + \|\nabla u\|_{p,\Omega}$;

$$W_p^2(\Omega) := \{u \in W_p^1(\Omega), \nabla^2 u \in L^p(\Omega)\},$$

with the norm $\|u\|_{W_p^2(\Omega)} = \|u\|_{W_p^1(\Omega)} + \|\nabla^2 u\|_{p,\Omega}$; and the following notation:

$$\begin{aligned} W &:= \overset{\circ}{W}_{2q}^2(\Omega); \\ \mathcal{V} &:= L^{2q}(\Omega); \\ H &:= L^{2q}(\Omega) \times \overset{\circ}{W}_{2q}^{2-\frac{1}{q}}(\Omega). \end{aligned}$$

3 Main results

We define the following nonlinear operator corresponding to (4):

$$\begin{aligned} F : W \times W \times \mathcal{V} &\longrightarrow H \times H \\ (u^n, p^n, f^n) &\longrightarrow F(u^n, p^n, f^n), \end{aligned}$$

where

$$F(u^n, p^n, f^n) = \left(\frac{u^{n+1} - u^n}{\tau} - \Delta \varphi(u^n) - \operatorname{div}(g(u^n) \nabla p^n), \gamma_0 u^n - u_0 \right), \left(\frac{u^{n+1} - u^n}{\tau} - \operatorname{div}(d(u^n) \nabla p^n) - f^n, \gamma_0 p^n - p_0 \right),$$

γ_0 being the trace operator $\gamma_0 u^n = u|_{t=0}$. Our hypotheses ensure that F is well defined.

3.1 Gâteaux differentiability

Theorem 1. *In addition to the hypotheses (H1) and (H2), let us suppose that (H3) $|\varphi'''| \leq c$.*

Then, the operator F is Gâteaux differentiable and for all $(e, w, h) \in W \times W \times \Upsilon$ its derivative is given by

$$\begin{aligned} \delta F(u^n, p^n, f^n)(e, w, h) &= \frac{d}{ds} F(u^n + se, p^n + sw, f^n + sh) \big|_{s=0} \\ &= (\delta F_1, \delta F_2) = \begin{pmatrix} \xi_1, \xi_2 \\ \xi_3, \xi_4 \end{pmatrix}, \end{aligned}$$

$\xi_1 = e - \operatorname{div}(\varphi'(u^n)\nabla e) - \operatorname{div}(\varphi''(u^n)e\nabla u^n) - \operatorname{div}(g(u^n)\nabla w) - \operatorname{div}(g'(u^n)e\nabla p^n)$,
 $\xi_2 = \gamma_0 e$, $\xi_3 = w - \operatorname{div}(d(u^n)\nabla w) - \operatorname{div}(d'(u^n)e\nabla p^n) - h$, $\xi_4 = \gamma_0 w$.
 Furthermore, for any optimal solution $(\bar{u}^n, \bar{p}^n, \bar{f}^n)$ of the problem of minimizing (3) among all the functions (u^n, p^n, f^n) satisfying (4), the image of $\delta F(\bar{u}^n, \bar{p}^n, \bar{f}^n)$ is equal to $H \times H$.

To prove Theorem 1 we make use of the following lemma.

Lemma 1. *The operator $\delta F(u^n, p^n, f^n) : W \times W \times \Upsilon \longrightarrow H \times H$ is linear and bounded.*

Proof (Lemma 1). For all $(e, w, h) \in W \times W \times \Upsilon$

$$\begin{aligned} \delta_{u^n} F_1(u^n, p^n, f^n)(e, w, h) &= e - \operatorname{div}(\varphi'(u^n)\nabla e) - \operatorname{div}(\varphi''(u^n)e\nabla u^n) \\ &\quad - \operatorname{div}(g(u^n)\nabla w) - \operatorname{div}(g'(u^n)e\nabla p^n) \\ &= e - \varphi'(u^n)\Delta e - \varphi''(u^n)\nabla u^n \cdot \nabla e - \varphi''(u^n)e\Delta u^n \\ &\quad - \varphi''(u^n)\nabla e \cdot \nabla u^n - \varphi'''(u^n)e|\nabla u^n|^2 - g(u^n)\Delta w - g'(u^n)\nabla u^n \cdot \nabla w \\ &\quad - g'(u^n)e\Delta p^n - g'(u^n)\nabla e \cdot \nabla p^n - g''(u^n)e\nabla u^n \cdot \nabla p^n, \end{aligned}$$

where $\delta_{u^n} F$ is the Gâteaux derivative of F with respect to u^n . Using our hypotheses we have

$$\begin{aligned} \|g''(u^n)e\nabla u^n \cdot \nabla p^n\|_{2q, \Omega} &\leq \|e\|_{\infty, \Omega} \|\nabla u^n \cdot \nabla p^n\|_{2q, \Omega} \\ &\leq \|e\|_{\infty, \Omega} \|\nabla u^n\|_{\frac{4q}{2-q}, \Omega} \|\nabla p^n\|_{4, \Omega} \\ &\leq c \|u^n\|_W \|p^n\|_W \|e\|_W. \end{aligned}$$

Evaluating each term of $\delta_{u^n} F_1$, we obtain

$$\begin{aligned} \|\delta_{u^n} F_1(u^n, p^n, f^n)(e, w, h)\|_{2q, Q_T} &\leq c (\|u^n\|_W, \|p^n\|_W, \|f^n\|_r) (\|e\|_W + \|w\|_W + \|h\|_r). \quad (5) \end{aligned}$$

In a similar way, we have for all $(e, w, h) \in W \times W \times \mathcal{Y}$ that

$$\begin{aligned}\delta_{p^n} F_2(u^n, p^n, f^n)(e, w, h) &= w - \operatorname{div}(d(u^n)\nabla w) - \operatorname{div}(d'(u^n)e\nabla p^n) - h \\ &= w - d(u^n)\Delta w - d'(u^n)\nabla u^n \cdot \nabla w - d'(u^n)e\Delta p^n \\ &\quad - d'(u^n)\nabla e \cdot \nabla u^n - d'(u^n)e\nabla u^n \cdot \nabla p^n - h,\end{aligned}$$

with $\delta_{p^n} F$ the Gâteaux derivative of F with respect to p^n . Then, using again our hypotheses, we obtain that

$$\begin{aligned}\|\delta_{p^n} F_2(u^n, p^n, f^n)(e, w, h)\|_{2q, \Omega} &\leq \|w\|_{2q, \Omega} + \|\nabla w\|_{2q, \Omega} + c\|\Delta w\|_{2q, \Omega} \\ &\quad + c\|\nabla u^n \cdot \nabla w\|_{2q, \Omega} + c\|e\Delta p^n\|_{2q, \Omega} \\ &\quad + c\|\nabla e \cdot \nabla u^n\|_{2q, \Omega} + c\|e\nabla u^n \cdot \nabla p^n\|_{2q, \Omega} + \|h\|_{2q, \Omega}.\end{aligned}\quad (6)$$

Applying similar arguments to all terms of (6), we then have

$$\begin{aligned}\|\delta_{p^n} F_2(u^n, p^n, f^n)(e, w, h)\|_{2q, \Omega} \\ \leq c(\|u^n\|_W, \|p^n\|_W, \|f^n\|_r)(\|e\|_W + \|w\|_W + \|h\|_r).\end{aligned}\quad (7)$$

Consequently, by (5) and (7) we can write

$$\begin{aligned}\|\delta F(u^n, p^n, f^n)(e, w, h)\|_{H \times H \times \mathcal{Y}} \\ \leq c(\|u^n\|_W, \|p^n\|_W, \|f^n\|_r)(\|e\|_W + \|w\|_W + \|h\|_r).\end{aligned}$$

□

Proof (Theorem 1). In order to show that the image of $\delta F(\bar{u}, \bar{p}, \bar{f})$ is equal to $H \times H$, we need to prove that there exists $(e, w, h) \in W \times W \times \mathcal{Y}$ such that

$$\begin{aligned}e - \operatorname{div}(\varphi'(\bar{u}^n)\nabla e) - \operatorname{div}(\varphi''(\bar{u}^n)e\nabla \bar{u}^n) \\ - \operatorname{div}(g(\bar{u}^n)\nabla w) - \operatorname{div}(g'(\bar{u}^n)e\nabla \bar{p}^n) = \alpha, \\ w - \operatorname{div}(d(\bar{u}^n)\nabla w) - \operatorname{div}(d'(\bar{u}^n)e\nabla \bar{p}^n) - h = \beta, \\ e|_{\partial\Omega} = 0, \quad e|_{t=0} = b, \\ w|_{\partial\Omega} = 0, \quad w|_{t=0} = a,\end{aligned}\quad (8)$$

for any (α, a) and $(\beta, b) \in H$. Writing the system (8) for $h = 0$ as

$$\begin{aligned}e - \varphi'(\bar{u}^n)\Delta e - 2\varphi''(\bar{u}^n)\nabla \bar{u}^n \cdot \nabla e - \varphi''(\bar{u}^n)e\Delta \bar{u}^n - \varphi'''(\bar{u}^n)e|\nabla \bar{u}^n|^2, \\ - g(\bar{u}^n)\Delta w - g'(\bar{u}^n)\nabla \bar{u}^n \cdot \nabla w - g'(\bar{u}^n)e\Delta \bar{p}^n \\ - g'(\bar{u}^n)\nabla \bar{p}^n \cdot \nabla e - g''(\bar{u}^n)e\nabla \bar{u}^n \cdot \nabla \bar{p}^n = \alpha, \\ w - d(\bar{u}^n)\Delta w - d'(\bar{u}^n)\nabla \bar{u}^n \cdot \nabla w - d'(\bar{u}^n)e\Delta \bar{p}^n \\ - d'(\bar{u}^n)\nabla \bar{u}^n \cdot \nabla \bar{e} - d'(\bar{u}^n)e\nabla \bar{u}^n \cdot \nabla \bar{p}^n = \beta, \\ e|_{\partial\Omega} = 0, \quad e|_{t=0} = b, \\ w|_{\partial\Omega} = 0, \quad w|_{t=0} = a,\end{aligned}\quad (9)$$

it follows from the regularity of the optimal solution that $\varphi''(\overline{u^n})\Delta\overline{u^n}$, $\varphi'''(\overline{u^n})|\nabla\overline{u^n}|^2$, $g'(\overline{u^n})\Delta\overline{p^n}$, $g''(\overline{u^n})\nabla\overline{u^n}.\nabla\overline{p^n}$, $d'(\overline{u^n})\Delta\overline{p^n}$, and $d''(\overline{u^n})\nabla\overline{u^n}.\nabla\overline{p^n}$ belong to $L^{2q_0}(\Omega)$; $\varphi''(\overline{u^n})\nabla\overline{u^n}$, $g'(\overline{u^n})\nabla\overline{u^n}$, $g'(\overline{u^n})\nabla\overline{p^n}$, and $d'(\overline{u^n})\nabla\overline{u^n}$ belong to $L^{4q_0}(\Omega)$. This ensures, in view of the results of [4, 6], existence of a unique solution of the system (9). Hence, there exists a $(e, w, 0)$ verifying (8). We conclude that the image of δF is equal to $H \times H$. \square

3.2 Necessary optimality condition

We consider the cost functional $J : W \times W \times \mathcal{Y} \rightarrow \mathbb{R}$ (3) and the Lagrangian \mathcal{L} defined by

$$\mathcal{L}(u^n, p^n, f^n, p_1, e_1, a, b) = J(u^n, p^n, f^n) + \left\langle F(u^n, p^n, f^n), \begin{pmatrix} p_1 & a \\ e_1 & b \end{pmatrix} \right\rangle,$$

where the bracket $\langle \cdot, \cdot \rangle$ denotes the duality between H and H' .

Theorem 2. *Under hypotheses (H1)–(H3), if $(\overline{u^n}, \overline{p^n}, \overline{f^n})$ is an optimal solution to the problem of minimizing (3) subject to (4), then there exist functions $(\overline{e_1}, \overline{p_1}) \in W_2^2(\Omega) \times W_2^2(\Omega)$ satisfying the following conditions:*

$$\begin{aligned} \overline{e_1} + \operatorname{div}(\varphi'(\overline{u^n})\nabla e_1) - d'(\overline{u^n})\nabla\overline{p^n}.\nabla\overline{p_1} - \varphi''(\overline{u^n})\nabla\overline{u^n}.\nabla\overline{e_1} \\ - g'(\overline{u^n})\nabla\overline{p^n}.\nabla\overline{e_1} &= \tau \sum_{n=1}^N (\overline{u^n} - U), \\ \overline{e_1}|_{\partial\Omega} &= 0, \quad \overline{e_1}|_{t=T} = 0, \\ \overline{p_1} + \operatorname{div}(d(\overline{u^n})\nabla\overline{p_1}) + \operatorname{div}(g(\overline{u^n})\nabla\overline{e_1}) &= \tau \sum_{n=1}^N (\overline{p^n} - P), \\ \overline{p_1}|_{\partial\Omega} &= 0, \quad \overline{p_1}|_{t=T} = 0, \\ q_0\beta_1\tau \sum_{n=1}^N |\overline{f^n}|^{2q_0-2}\overline{f^n} &= \overline{p_1}. \end{aligned} \tag{10}$$

Proof. Let $(\overline{u^n}, \overline{p^n}, \overline{f^n})$ be an optimal solution to the problem of minimizing (3) subject to (4). It is well known (cf. e.g. [2]) that there exist Lagrange multipliers $((\overline{p_1}, \overline{a}), (\overline{e_1}, \overline{b})) \in H' \times H'$ verifying

$$\delta_{(u^n, p^n, f^n)} \mathcal{L}(\overline{u^n}, \overline{p^n}, \overline{f^n}, \overline{p_1}, \overline{e_1}, \overline{a}, \overline{b})(e, w, h) = 0 \quad \forall (e, w, h) \in W \times W \times \mathcal{Y},$$

with $\delta_{(u^n, p^n, f^n)} \mathcal{L}$ the Gâteaux derivative of \mathcal{L} with respect to (u^n, p^n, f^n) . This leads to the following system:

$$\begin{aligned}
& \tau \sum_{n=1}^N \int_{\Omega} ((\bar{u}^n - U)e + (\bar{p}^n - P)w + q_0 \beta_1 |\bar{f}^n|^{2q_0-2} \bar{f}^n h) dx \\
& - \int_{\Omega} \left((e - \operatorname{div}(\varphi'(\bar{u}^n) \nabla e) - \operatorname{div}(\varphi''(\bar{u}^n) e \nabla \bar{u}^n) \right. \\
& \quad \left. - \operatorname{div}(g(\bar{u}^n) \nabla w) - \operatorname{div}(g'(\bar{u}^n) e \nabla \bar{p}^n)) \bar{e}_1 \right) dx \\
& - \int_{\Omega} (w - \operatorname{div}(d(\bar{u}^n) \nabla w) - \operatorname{div}(d'(\bar{u}^n) e \nabla \bar{p}^n) - h) \bar{p}_1 dx \\
& - \langle \gamma_0 e, \bar{a} \rangle + \langle \gamma_0 w, \bar{b} \rangle = 0 \quad \forall (e, w, h) \in W \times W \times \mathcal{Y}.
\end{aligned}$$

The above system is equivalent to the following one:

$$\begin{aligned}
& \int_{\Omega} \left(\tau \sum_{n=1}^N (\bar{u}^n - U)e - \operatorname{div}(d'(\bar{u}^n) e \nabla \bar{p}^n) \bar{p}_1 + e \bar{e}_1 - \operatorname{div}(\varphi'(\bar{u}^n) \nabla e) \bar{e}_1 \right. \\
& \quad \left. - \operatorname{div}(\varphi''(\bar{u}^n) e \nabla \bar{u}^n) \bar{e}_1 - \operatorname{div}(g'(\bar{u}^n) e \nabla \bar{p}^n) \bar{e}_1 \right) dx \\
& + \int_{\Omega} \left(\tau \sum_{n=1}^N (\bar{p}^n - P)w + w \bar{p}_1 - \operatorname{div}(d(\bar{u}^n) \nabla w) \bar{p}_1 - \operatorname{div}(g(\bar{u}^n) \nabla w) \bar{e}_1 \right) dx \\
& \quad + \int_{\Omega} \left(q_0 \beta_1 \tau \sum_{n=1}^N |\bar{f}^n|^{2q_0-2} \bar{f}^n h - \bar{p}_1 h \right) dx \\
& + \langle \gamma_0 e, \bar{a} \rangle + \langle \gamma_0 w, \bar{b} \rangle = 0 \quad \forall (e, w, h) \in W \times W \times \mathcal{Y}.
\end{aligned} \tag{11}$$

In others words, we have

$$\begin{aligned}
& \int_{\Omega} \left(\tau \sum_{n=1}^N (\bar{u}^n - U) + d'(u) \nabla \bar{p}^n \cdot \nabla \bar{p}_1 - \bar{e}_1 - \operatorname{div}(\varphi'(\bar{u}^n) \nabla \bar{e}_1) \right. \\
& \quad \left. + \varphi''(\bar{u}^n) \nabla \bar{u}^n \cdot \nabla \bar{e}_1 + g'(\bar{u}^n) \nabla \bar{p}^n \cdot \nabla \bar{e}_1 \right) e dx \\
& + \int_{\Omega} \left(\tau \sum_{n=1}^N (\bar{p}^n - P) + \bar{p}_1 - \operatorname{div}(d(\bar{u}^n) \nabla \bar{p}_1) - \operatorname{div}(g(\bar{u}^n) \nabla \bar{e}_1) \right) w dx \\
& \quad + \int_{\Omega} \left(q_0 \beta_1 \tau \sum_{n=1}^N |\bar{f}^n|^{2q_0-2} \bar{f}^n h - \bar{p}_1 h \right) dx \\
& + \langle \gamma_0 e, \bar{a} \rangle + \langle \gamma_0 w, \bar{b} \rangle = 0 \quad \forall (e, w, h) \in W \times W \times \mathcal{Y}.
\end{aligned} \tag{12}$$

Consider now the system

$$\begin{aligned}
e_1 + \operatorname{div}(\varphi'(\overline{u^n})\nabla e_1) - d'(\overline{u^n})\nabla\overline{p^n}\cdot\nabla p_1 - \varphi''(\overline{u^n})\nabla\overline{u^n}\cdot\nabla e_1 \\
- g'(\overline{u^n})\nabla\overline{p^n}\cdot\nabla e_1 = \tau \sum_{n=1}^N (\overline{u^n} - U), \\
p_1 + \operatorname{div}(d(\overline{u^n})\nabla p_1) + \operatorname{div}(g(\overline{u^n})\nabla e_1) = \tau \sum_{n=1}^N (\overline{p^n} - P), \\
e_1|_{\partial\Omega} = p_1|_{\partial\Omega} = 0, \quad e_1|_{t=T} = p_1|_{t=T} = 0,
\end{aligned} \tag{13}$$

with unknowns (e_1, p_1) which is uniquely solvable in $W_2^2(\Omega) \times W_2^2(\Omega)$ by the theory of elliptic equations [4]. The problem of finding $(e, w) \in W \times W$ satisfying

$$\begin{aligned}
e - \operatorname{div}(\varphi'(\overline{u^n})\nabla e) - \operatorname{div}(\varphi''(\overline{u^n})e\nabla\overline{u^n}) - \operatorname{div}(g(\overline{u^n})\nabla w) \\
- \operatorname{div}(g'(\overline{u^n})e\nabla\overline{p^n}) = \operatorname{sign}(e_1 - \overline{e_1}), \\
w - \operatorname{div}(d(\overline{u^n})\nabla w) - \operatorname{div}(d'(\overline{u^n})e\nabla\overline{p^n}) = \operatorname{sign}(p_1 - \overline{p_1}), \\
\gamma_0 e = \gamma_0 w = 0,
\end{aligned} \tag{14}$$

is also uniquely solvable on $W_{2q}^2(\Omega) \times W_{2q}^2(\Omega)$. Let us choose $h = 0$ in (12) and multiply (13) by (e, w) . Then, integrating by parts and making the difference with (12) we obtain:

$$\begin{aligned}
& \int_{\Omega} \left(e - \operatorname{div}(\varphi'(\overline{u^n})\nabla e) - \operatorname{div}(\varphi''(\overline{u^n})e\nabla\overline{u^n}) - \operatorname{div}(g(\overline{u^n})\nabla w) \right. \\
& \quad \left. - \operatorname{div}(g'(\overline{u^n})e\nabla\overline{p^n}) \right) (e_1 - \overline{e_1}) dx \\
& + \int_{\Omega} (w - \operatorname{div}(d(\overline{u^n})\nabla w) - \operatorname{div}(d'(\overline{u^n})e\nabla\overline{p^n})) (p_1 - \overline{p_1}) dx \\
& + \langle \gamma_0 e, \gamma_0 \overline{e_1} - \overline{a} \rangle + \langle \gamma_0 w, \gamma_0 \overline{p_1} - \overline{b} \rangle = 0 \quad \forall (e, w) \in W \times W.
\end{aligned} \tag{15}$$

Choosing (e, w) in (15) as the solution of the system (14), we have

$$\int_{\Omega} \operatorname{sign}(e_1 - \overline{e_1})(e_1 - \overline{e_1}) dx dt + \int_{\Omega} \operatorname{sign}(p_1 - \overline{p_1})(p_1 - \overline{p_1}) dx = 0.$$

It follows that $e_1 = \overline{e_1}$ and $p_1 = \overline{p_1}$. Coming back to (15), we obtain $\gamma_0 \overline{e_1} = \overline{a}$ and $\gamma_0 \overline{p_1} = \overline{b}$. On the other hand, choosing $(e, w) = (0, 0)$ in (12), we get

$$\int_{\Omega} \left(\beta_1 \tau \sum_{n=1}^N |\overline{f^n}|^{2q_0-2} \overline{f^n} - \overline{p_1} \right) h dx = 0, \quad \forall h \in \mathcal{X}.$$

Then (10) follows, which concludes the proof of Theorem 2. \square

We claim that the results we obtain here are useful for numerical implementations. This is still under investigation and will be addressed in a forthcoming publication.

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